$d\eta / dt \leqslant -1$ and imposing certain other restrictions on the function $\eta[t]$. However, this complicates determination of the generalized solution $y^*[t]$ of Eq. (1.1).

BIBLIOGRAPHY

1. Isaacs, R., Differential Games. Moscow, "Mir", 1967.
2. Pontriagin, L. S., On the theory of differential games. Uspekhi Mat. Nauk Vol. 21, №4, 1966.
3. Pshenichnyi, B. N., On the pursuit problem. Kibernetika №6, 1967.
4. Petrosian, L. A., A dynamic pursuit game with friction forces acting. Dokl. Akad. Nauk ArmSSR Vol. 44, №1, 1967.
5. Pozharitskii, G. K., Impulsive tracking in the case of second-order monotype linear objects. PMM Vol. 30, №5, 1966.
6. Krasovskii N. N., On a problem of tracking. PMM Vol. 27, №2, 1963.
7. Krasovskii, N. N., Theory of Motion Control. Linear Systems. Moscow, "Nauka", 1968.
8. Krasovskii, N. N., A certain peculiarity of the game encounter of motion. Differentsial'nye Uravneniia Vol. 4, № 5, 1968.
9. Subbotin, A. I., Regularizing a certain problem on the encounter of motions. Differentsial'nye Uravneniia Vol. 4, № 5, 1968.
10. Krasovskii, N. N. and Subbotin, A. I., The problem of convergence of controlled objects. PMM Vol. 32, № 4, 1968.
11. Karlin, S., Mathematical Methods in Games Theory, Programing and Economics. Moscow, "Mir", 1964
12. Pontriagin, L. S., Boltianskii, V. G., Gamkrelidze, R. V. and Mishchenko, E. F., The Mathematical Theory of Controlled Processes. Moscow, Fizmatgiz, 1961.
13. Filippov, A. F., Differential equations with a discontinuous game part. Mat. Sb., Vol. 51(93), №1, 1959.

Translated by A. Y.

STABILITY OF MOTION OVER A FINITE TIME INTERVAL

PMM Vol. 32, №6, 1968, pp. 977-986

K. A. ABGARIAN
(Moscow)

(Received May 13, 1968)

A family of necessary and sufficient conditions for the stability and instability of motion over a finite time interval is constructed. This is made possible by a generalization of Kamenkov's formulation of the problem of stability over a finite time interval.

1. In his investigation of mechanical systems whose perturbed motion is described by the equations

$$\frac{dx_i}{dt} = X_i(t;\ x_1, \ldots, x_n) \qquad (i = 1, \ldots, n) \tag{1.1}$$

where X_i are real functions of real variables which vanish for $x_i = 0$ $(i = 1, \ldots, n)$ and can be expanded in series in whole nonnegative powers of x_i in the neighborhood of the origin $(x_i = 0)$, Kamenkov introduced the following definition of stability of motion over a finite time interval [1].

If the differential equations of perturbed motion (1.1) are such that for a sufficiently small positive A the quantities x_s considered as functions of time satisfy the condition

$$\sum_{s=1}^{n} (a_{s1}x_1 + \cdots + a_{sn}x_n)^2 \leqslant A \tag{1.2}$$

over a finite time interval $[t_0,\ t_0 + \Delta t]$ provided the initial values x_{i0} of these functions satisfy the condition

$$\sum_{s=1}^{n} (a_{s1}x_{10} + \cdots + a_{sn}x_{n0})^2 \leqslant A \quad (\det (a_{ij}) \neq 0) \tag{1.3}$$

then the unperturbed motion is stable over the time interval Δt; otherwise the motion is unstable, i.e. $\Delta t = 0$.

On the basis of the above definition Kamenkov obtained the conditions of stability and instability of unperturbed motion in the first approximation. According to these conditions the problem of stability of motion in noncritical cases is resolved by the signs of the real parts of the roots of the characteristic polynomial of the first-approximation equations at the initial instant t_0.

Instead of the constant domain of limiting deviations (1.2), Lebedev [2 and 3] used the fixed-sign function $V\ (t;\ x_1, \ldots, x_n)$ which depends explicitly on time to introduce the variable domain

$$V\ (t;\ x_1, \ldots, x_n) \leqslant A \tag{1.4}$$

In this way he obtained sufficient conditions of stability which take account of the character of variation of the coefficients in the equations of perturbed motion with respect to the time t. This entails a rigid restriction on the diameter of the domain, i.e. on the upper bound of the disturbances between any two points of the domain; considerable leeway still remains in the choice of the remaining dimensions.

We shall introduce necessary and sufficient conditions of stability and instability of motion in the following formulation.

Definition. If the equations of perturbed motion are such that for a sufficiently small $\rho > 0$ any solution $x\ (t)$ of the equations whose initial value $x_0 = x\ (t_0)$ conforms to the inequality

$$(G\ (t_0)\ x_0,\ G\ (t_0)\ x_0) \leqslant \rho \tag{1.5}$$

satisfies the condition

$$(G\ (t)\ x,\ G\ (t)\ x) \leqslant \rho \tag{1.6}$$

(where $G\ (t)$ is a given bounded matrix) over some finite interval $[t_0,\ t_0 + \Delta t]$, then the unperturbed motion is stable relative to domain (1.6) over the interval $[t_0,\ t_0{}' + \Delta t]$; otherwise it is unstable, i.e. $\Delta t = 0$.

The domain of limiting deviations $x_s\ (s = 1,\ldots,\ n)$ (i.e. of the elements of the column matrix x) is given here by means of a nonnegative function,

$$V\ (t,\ x) \equiv (G\ (t)\ x,\ G\ (t)\ x)$$

The stability of motion relative to domain (1.6) will be called "uniform" over the interval $[t_1,\ t_2)$ if the unperturbed motion is stable for all $t_0 \in [t_1,\ t_2)$.

2. Let the equations of perturbed motion in vector-matrix notation be written as

$$dx\ /\ dt = U\ (t)\ x + H\ (t,\ x) \tag{2.1}$$

where U is a square matrix of order n, and x and H are column matrices. The elements of the matrix H (i.e. the nonlinear functions of the deviations x_s) are such that

$$\lim_{x\to 0}\frac{H(t,x)}{\|x\|}=0 \tag{2.2}$$

uniformly over t over some interval $[t_0, T]$

Let us assume that the matrix U is differentiable with respect to t and that it is of simple structure. There then exists a nondegenerate and differentiable matrix K which transforms the matrix U into diagonal form

$$K^{-1}(t)\,U(t)\,K(t)=\Lambda(t)\equiv \mathrm{diag}\,(\lambda_1(t),\ldots,\lambda_n(t)) \tag{2.3}$$

The diagonal elements of the matrix Λ are the eigenvalues of the matrix U, and the columns of the matrix K are its eigenvectors.

Assuming that the coloumns K_σ $(\sigma=1,\ldots,n)$ of the matrix K are normalized (e.g. that their Euclidean norm is equal to unity) and setting

$$V(t,x)\equiv(K^{-1}(t)\,x,\ K^{-1}(t)\,x) \tag{2.4}$$

we define the domain of limiting deviations as

$$(K^{-1}(t)\,x,\ K^{-1}(t)\,x)\leqslant\rho \tag{2.5}$$

Geomertically, domain (2.5) is an n-dimensional ellipsoid bounded by the surface

$$(K^{-1}(t)\,x,\ K^{-1}(t)x)=\rho \tag{2.6}$$

Each of the $2n$ rays

$$x=\pm K_\sigma(t)\,s \qquad (\sigma=1,\ldots,n;\ 0<s<\infty)$$

intersects surface (2.6) once for the parameter value $s=\sqrt{\rho}$. The points of intersection lie at a constant distance $\sqrt{\rho}$ from the origin $(x=0)$. In fact,

$$(K^{-1}(t)\,K_\sigma(t)\sqrt{\rho},\ K^{-1}(t)\,K_\sigma(t)\sqrt{\rho})=\rho\sum_{i=1}^{n}\delta_{i\sigma}^2=\rho \qquad \left(\delta_{ij}=\begin{cases}1 & (i=j)\\ 0 & (i\neq j)\end{cases}\right)$$

$$\|K_\sigma(t)\sqrt{\rho}\|=\|K_\sigma(t)\|\sqrt{\rho}=\sqrt{\rho}$$

Let us investigate the conditions of stability and instability of unperturbed motion relative to domain (2.5). Setting

$$x=K(t)\,y \tag{2.7}$$

we have

$$V(t,x)=(y,y)=\|y\|^2 \tag{2.8}$$

Hence,

$$\frac{dV}{dt}=2\|y\|\frac{d\|y\|}{dt} \tag{2.9}$$

In the new variables the equations of perturbed motion become

$$\frac{dy}{dt}=\Lambda(t)\,y-K^{-1}(t)\frac{dK(t)}{dt}\,y+K^{-1}(t)\,H(t,Ky) \tag{2.10}$$

From (2.10) we find that

$$\frac{d\|y\|}{dt}=\sum_{\sigma=1}^{n}\mathrm{Re}\,\lambda_\sigma\frac{|y_\sigma|^2}{\|y\|}+\frac{y^*Py}{\|y\|}+\frac{1}{2\|y\|}(y^*K^{-1}H+H^*K^{*^{-1}}y) \tag{2.11}$$

Here

$$P=-\frac{1}{2}\left(K^{-1}\frac{dK}{dt}+\frac{dK^*}{dt}K^{*^{-1}}\right)$$

and y_σ $(\sigma=1,\ldots,n)$ are the elements of the column matrix y.

From Expressions (2.9) and (2.11), we find that the derivative of positive-definite function (2.4) with respect to t computed by way of the equations of perturbed motion is given by

$$\frac{1}{2}\frac{dV}{dt} = \sum_{\sigma=1}^{n} \operatorname{Re}\lambda_\sigma |y_\sigma|^2 + y^* P y + \frac{1}{2}(y^* K^{-1} H + H^* K^{*-1} y) \qquad (2.12)$$

Let us set

$$\mu(t) = \max_\sigma (\operatorname{Re}\lambda_\sigma(t))$$

By $\nu_{\min}(t)$ and $\nu_{\max}(t)$ we denote the minimum and maximum eigenvalues, respectively, of the Hermitian matrix P.

As we know,

$$\nu_{\min}\|y\|^2 \leqslant y^* P y \leqslant \nu_{\max}\|y\|^2 \qquad (2.13)$$

Theorem 2.1. If

$$\mu(t_0) + \nu_{\max}(t_0) < 0 \qquad (2.14)$$

then unperturbed motion (the trivial solution of Eq. (2.1)) is stable over the finite interval $[t_0, t_0 + \Delta t]$.

Proof.

$$\frac{H(t, Ky)}{\|y\|} \to 0 \quad \text{as } y \to 0 \qquad (2.15)$$

uniformly in t over the segment $[0, T]$.

In fact,

$$\frac{H(t, Ky)}{\|y\|} = \frac{H(t, x)\|K\|}{\|K\|\|y\|} \leqslant \frac{H(t, x)}{\|x\|}\|K\| \to 0 \quad \text{as } y \to 0$$

by virtue of condition (2.2), since $\|K\|$ is a bounded quantity and since

$$\|x\| \leqslant \|K\|\|y\| \to 0 \quad \text{as } y \to 0$$

Taking account of (2.13) and (2.15), we find from (2.12) that

$$\frac{1}{2}\frac{dV}{dt} \leqslant (\mu(t) + \nu_{\max}(t))\|y\|^2 + o(\|y\|^2)$$

From this we see that if inequality (2.14) holds, then for sufficiently small $\|y\|$ at the point $t = t_0$ (and, by continuity, within some finite interval $[t_0, t_0 + \Delta t] \subset [t_0, T]$), we find that $dV/dt < 0$, which proves the Theorem.

Theorem 2.2. If

$$\mu(t_0) + \nu_{\min}(t_0) > 0 \qquad (2.16)$$

then unperturbed motion (the trivial solution of Eq. (2.1)) is not stable over the finite time interval $[t_0, t_0 + \Delta t]$, i.e. $\Delta t = 0$.

Proof. Let us integrate (2.11). We obtain

$$\|y(t)\| = \|y(t_0)\| \exp\int_{t_0}^{t}\left[\sum_{\sigma=1}^{n} \operatorname{Re}\lambda_\sigma \frac{|y_\sigma|^2}{\|y\|^2} + \frac{y^* P y}{\|y\|^2} + O(\|y\|)\right] dt \qquad (2.17)$$

If

$$\varphi(t, y) \equiv \sum_{\sigma=1}^{n} \operatorname{Re}\lambda_\sigma \frac{|y_\sigma|^2}{\|y\|^2} + \frac{y^* P y}{\|y\|^2} \neq 0$$

then for sufficiently small $\|y\|$ the sign of the integrand is the same as that of the function $\varphi(t, y)$.

Let us assume that

$$\mu(t_0) = \operatorname{Re}\lambda_s(t_0)$$

Let us consider the particular solution $x^\circ = K(t) y^\circ$ of Eq. (2.1) as determined by the initial conditions

$$y_s(t_0) = \sqrt{\rho}, \quad y_\sigma(t_0) = 0 \qquad (\sigma \neq s) \qquad (2.18)$$

By virtue of (2.13), (2.16) and (2.18) we have

$$\varphi(t_0, y^\circ(t_0)) \geqslant \mu(t_0) + \nu_{\min}(t_0) > 0$$

Hence, for sufficiently small ρ at the point t_0 the integrand function in Eq. (2.17) is positive. By continuity, it is also positive in some neighborhood of this point. Hence, in this neighborhood we have $\frac{dV(t, x^\circ)}{dt} = 2\|y^\circ\|\frac{d\|y^\circ\|}{dt} > 0$

Thus, if inequality (2.16) is valid, then there are particular solutions along which in the neighborhood of the point t_0 we have

$$V(t, x(t)) > V(t_0, x(t_0)) \quad (t > t_0)$$

which means that condition (1.6) is not fulfilled. The Theorem has been proved.

Theorem 2.3. If

$$\mu(t_0) + \nu_{\min}(t_0) \leqslant 0 \leqslant \mu(t_0) + \nu_{\max}(t_0) \tag{2.19}$$

then unperturbed motion (the trivial solution of Eq. (2.1)) may not be stable over a finite time interval.

Proof. Relations (2.19) and (2.13) admit of the existence of a particular solution $x^\circ = Ky^\circ$ which satisfies the equations

$$\varphi(t_0, y^\circ(t_0)) = 0, \qquad \|y^\circ(t_0)\| = \sqrt{\rho}$$

The sign of the integrand in Eq. (2.17) for this solution is determined by the sign of $O(\|y^\circ\|)$, so that depending on the properties of the nonlinear terms for $t = t_0$ (and, by continuity, within some neighborhood of the point t_0) the integrand may be positive. Hence, in this neighborhood $dV(t, x^\circ) / dt > 0$, which means that inequality (1.6) is not fulfilled.

Therefore,

$$\begin{aligned} &\mu(t_0) + \nu_{\max}(t_0) < 0 && \text{is a sufficient condition for stability} \\ &\mu(t_0) + \nu_{\min}(t_0) \leqslant 0 && \text{is the necessary condition for stability} \\ &\mu(t_0) + \nu_{\min}(t_0) > 0 && \text{is a sufficient condition for instability} \\ &\mu(t_0) + \nu_{\max}(t_0) \geqslant 0 && \text{is the necessary condition for instability} \end{aligned} \tag{2.20}$$

In the special case where $U = \text{const}$, we have $\nu_{\max,\min} = 0$, since $K = \text{const}$ and $P = 0$, and since the conditions (2.20) coincide with the corresponding conditions for stability and instability obtained by Kamenkov.

The disposition of the stability and instability domains in the plane of the eigenvalue λ as determined by the signs of the eigenvalues $\nu_{\max}$ and $\nu_{\min}$ is shown for the general case in Fig. 1 (the stability domains are marked with a plus sign, the instability domains with a minus sign). We see that the motion may be stable even when the eigenvalues of the matrix U includes values with positive real parts (Fig. 1c); conversely, unperturbed motion may turn out to be unstable even when the real parts of all the eigenvalues are negative (Fig. 1a).

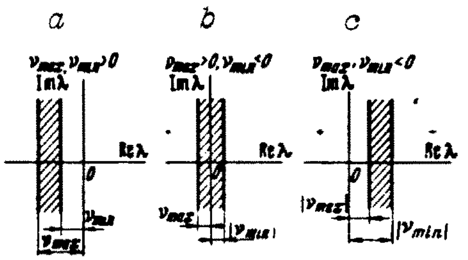

Fig. 1

Conditions (2.20) do not solve the problem of stability if

$$-\nu_{\min} > \mu > -\nu_{\max} \tag{2.21}$$

The corresponding strips in Figs. 1a, 1b and 1c are shaded.

In the next Section we shall describe a method for constructing a family of conditions similar to (2.20) whith whose aid the "strip of uncertainty" (2.21) can be narrowed substantially, especially in those cases where the coefficients of the first-approximation equations are slowly varying functions of t.

3. Instead of (2.1) let us consider the more general equation

$$dx / dt = U(\tau) x + H(t, x) \tag{3.1}$$

where $\tau = \varepsilon t$ is the so-called "slow time" and ε is some real parameter. When $\varepsilon = 1$ Eqs. (2.1) and (3.1) coincide.

The nondegenerate transformation

$$x = K^{(m)}(\tau, \varepsilon) y \tag{3.2}$$

transforms Eq. (3.1) into

$$dy / dt = \Lambda^{(m)}(\tau, \varepsilon) y - K^{(m)-1}(\tau, \varepsilon) N^{(m)}(\tau, \varepsilon) y + K^{(m)-1}(\tau, \varepsilon) H(t, K^{(m)} y) \tag{3.3}$$

where

$$N^{(m)}(\tau, \varepsilon) = \varepsilon\, dK^{(m)}(\tau, \varepsilon)/d\tau - U(\tau) K^{(m)}(\tau, \varepsilon) + K^{(m)}(\tau, \varepsilon) \Lambda^{(m)}(\tau, \varepsilon) \tag{3.4}$$

Let us assume that $U(\tau)$ is a matrix which is differentiable l times on the segment $[0, L]$. Then, making use of the algorithm given in [4], we can construct a transformation (3.2) such that the matrix $\Lambda^{(m)}$ has a diagonal, or at least a quasidiagonal structure, and the matrix $N^{(m)}$ satisfies the condition

$$\lim_{\varepsilon \to 0} \frac{N^{(m)}(\tau, \varepsilon)}{\varepsilon^m} = 0 \qquad (m = 0, 1, 2, \ldots, l-1) \tag{3.5}$$

Let us limit ourselves to the case where U has only simple eigenvalues on $[0, L]$. Here the matrices $K^{(m)}$ and $\Lambda^{(m)}$ can be constructed in the form of finite sums

$$K^{(m)}(\tau, \varepsilon) = \sum_{k=0}^{m} \varepsilon^k K^{[k]}(\tau), \qquad \Lambda^{(m)}(\tau, \varepsilon) = \sum_{k=0}^{m} \varepsilon^k \Lambda^{[k]}(\tau) \tag{3.6}$$

$$K^{[k]} = (K_1^{[k]} \cdots K_n^{[k]}), \qquad \Lambda^{[k]} = \operatorname{diag}(\lambda_1^{[k]} \cdots \lambda_n^{[k]}) \tag{3.7}$$

Here $K_\sigma^{[k]}$ are column matrices and $\lambda_\sigma^{[k]}$ are scalar functions.

Let $K^{[k]}$ and $\Lambda^{[k]}$ $(k = 0, 1, \ldots, m)$ be such that

$$UK^{[0]} = K^{[0]}\Lambda^{[0]} \tag{3.8}$$

$$UK^{[k]} = K^{[k]}\Lambda^{[0]} + K^{[0]}\Lambda^{[k]} + D^{[k-1]} \qquad (k = 1, \ldots, m) \tag{3.9}$$

where

$$D^{[k-1]} = \sum_{\alpha=1}^{k-1} K^{[k-\alpha]}\Lambda^{[\alpha]} + \frac{dK^{[k-1]}}{d\tau} \tag{3.10}$$

It is now easy to verify that

$$N^{(m)} = \varepsilon^{m+1} \sum_{\nu=1}^{m} \sum_{\alpha=\nu}^{m} \varepsilon^{\nu-1} K^{[m-\alpha+\nu]}\Lambda^{[\alpha]} \tag{3.11}$$

and requirement (3.5) is fulfilled provided $K^{[j]}$, $\Lambda^{[j]}$ $(j = 1, \ldots, m)$ are bounded.

Equation (3.8) is satisfied identically if

$$K_\sigma^{[0]} \equiv K_\sigma, \quad \lambda_\sigma^{[0]} \equiv \lambda_\sigma \qquad (\sigma = 1, \ldots, n) \tag{3.12}$$

As in the previous section, we shall assume from now on that the Euclidean norm of the eigenvectors K_σ is equal to unity. By virtue of (3.7) and (3.10) we have

$$D^{[k-1]} = (D_1^{[k-1]} \dots D_n^{[k-1]}), \qquad D_\sigma^{[k-1]} = \sum_{\alpha=1}^{k-1} K_\sigma^{[k-\alpha]} \lambda_\sigma^{[\alpha]} + \frac{dK_\sigma^{[k-1]}}{d\tau} \tag{3.13}$$

Each equation of (3.9) breaks down into n independent matrix relations,

$$UK_\sigma^{[k]} = K_\sigma^{[k]}\lambda_\sigma + K_\sigma\lambda_\sigma^{[k]} + D_\sigma^{[k-1]} \qquad (\sigma = 1, \dots, n) \tag{3.14}$$

The general solution of Eqs. (3.14) is of the form

$$K_\sigma^{[k]} = P_\sigma D_\sigma^{[k-1]} + K_\sigma q_\sigma^{[k]}, \quad \lambda_\sigma^{[k]} = -M_\sigma D_\sigma^{[k-1]} \quad (\sigma = 1, \dots, n)$$

$$P_\sigma = \sum_{s \neq \sigma} \frac{K_s M_s}{\lambda_s - \lambda_\sigma} \tag{3.15}$$

Here M_s $(s = 1, \dots, n)$ are the rows of the matrix $M \equiv K^{-1}$, and $q_\sigma^{[k]}(\sigma = 1, \dots, n)$ are arbitraty bounded scalar functions of τ. This arbitrariness must be restricted by the condition of existence of derivatives with respect to τ of order up to $m - k + 1$, inclusive

The matrix $D_\sigma^{[k-1]}$ does not depend on $K_\sigma^{[r]}$, $\Lambda_\sigma^{[r]}$ $(r \geqslant k)$, so that in computing $K_\sigma^{[k]}$ and $\Lambda_\sigma^{[k]}$ we assume that this matrix is already known.

Formulas (3.15) are recurrent. They can be used to determine successively all of the terms of finite sums (3.6).

The arbitrariness involved in the construction of $K^{(m)}$ and $\Lambda^{(m)}$ can be used to normalize the columns of the matrix $K^{(m)}$. Since

$$K_\sigma^{(m)} = K_\sigma + \sum_{k=1}^{m} \varepsilon^k K_\sigma^{[k]}$$

it follows that the square of the norm of the column $K_\sigma^{(m)}$ of the matrix $K^{(m)}$ is

$$\| K_\sigma^{(m)} \|^2 = K_\sigma^* K_\sigma + \sum_{k=1}^{m} \sum_{\alpha=0}^{k} \varepsilon^k K_\sigma^{[k-\alpha]*} K_\sigma^{[\alpha]} + \varepsilon^{[m+1]} \sum_{k=1}^{m} \sum_{\alpha=k}^{m} \varepsilon^{k-1} K_\sigma^{[m-\alpha+k]*} K_\sigma^{[\alpha]} \tag{3.16}$$

The arbitrary functions $q_\sigma^{(k)}$ $(k = 1, \dots, m)$ can be chosen in such a way that the first double sum in Eq. (3.16) vanishes.

In fact,

$$\sum_{\alpha=0}^{k} K_\sigma^{[k-\alpha]*} K_\sigma^{[\alpha]} = K_\sigma^{[k]*} K_\sigma + K_\sigma^* K_\sigma^{[k]} + \sum_{\alpha=1}^{k-1} K_\sigma^{[k-\alpha]*} K_\sigma^{[\alpha]} =$$

$$= q_\sigma^{[k]*} + q_\sigma^{[k]} + D_\sigma^{[k-1]*} P_\sigma^* K_\sigma + K_\sigma^* P_\sigma D_\sigma^{[k-1]} + \sum_{\alpha=1}^{k-1} K_\sigma^{[k-\alpha]*} K_\sigma^{[\alpha]} = 0$$

if, for example

$$q_\sigma^{[k]} = - K_\sigma^* P_\sigma D_\sigma^{[k-1]} - \frac{1}{2} \sum_{\alpha=1}^{k-1} K_\sigma^{[k-\alpha]*} K_\sigma^{[\alpha]}$$

With $q_\sigma^{[k]}$ $(k = 1, \dots, m)$ the norm of the columns of the matrix $K^{(m)}$ is equal to unity to within quantities of the order ε^{m+1}.

Turning now to the establishment of the conditions of stability and instability of the trivial solution of Eq. (3.1), we define the domain of limiting deviations by the relation

$$(K^{(m)-1}(\tau, \varepsilon) x, K^{(m)-1}(\tau, \varepsilon) x) \leqslant \rho \tag{3.17}$$

Geometrically, domain (3.17) is an n-dimensional ellipsoid bounded by the surface

$$(K^{(m)-1}(\tau, \varepsilon)\, x,\ K^{(m)-1}(\tau, \varepsilon)\, x) = \rho \tag{3.18}$$

Each of the $2n$ rays

$$x = \pm K_s^{(m)}(\tau, \varepsilon)\, s \qquad (s = 1, \ldots, n;\ 0 < s < \infty)$$

intersects surface (3.18) for the value $s = \sqrt{\rho}$. To within quantities of the order ε^{m+1} the points of intersection lie at the constant distance $\sqrt{\rho}$ from the origin $(x = 0)$.

The conditions of stability and instability of the trivial solution of Eq. (3.1) relative to domain (3.17) are determined by the following theorems whose proof is entirely analogous to the proof of Theorems 2.1–2.3. Let

$$\mu^{(m)}(\tau, \varepsilon) = \max_s (\operatorname{Re} \lambda_s^{(m)}(\tau, \varepsilon)) \tag{3.19}$$

where

$$\lambda_s^{(m)}(\tau, \varepsilon) = \lambda_s(\tau) + \sum_{k=1}^{m} \varepsilon^k \lambda_s^{[k]}(\tau) \qquad (s = 1, \ldots, n)$$

are the diagonal elements of the diagonal matrix $\Lambda^{(m)}(\tau, \varepsilon)$;

$$\nu_{\min}^{(m)}(\tau, \varepsilon);\ \ \nu_{\max}^{(m)}(\tau, \varepsilon)$$

are, respectively, the minimum and maximum eigenvalues of the Hermitian matrix

$$P^{(m)} = -[K^{(m)-1}N^{(m)} + (K^{(m)-1}N^{(m)})^*]/2\varepsilon^{m+1}$$

By virtue of Eq. (3.11) the matrix $P^{(m)}$ is regular relative to ε in the neighborhood of the point $\varepsilon = 0$.

Theorem 3.1. If

$$\mu^{(m)}(\tau_0, \varepsilon) + \varepsilon^{m+1}\nu_{\max}^{(m)}(\tau_0, \varepsilon) < 0 \qquad (\tau_0 = \varepsilon t_0 \in [0, L])$$

then the trivial solution of Eq. (3.1) is stable on the finite interval $[t_0, t_0 + \Delta t]$.

Theorem 3.2. If

$$\mu^{(m)}(\tau_0, \varepsilon) + \varepsilon^{m+1}\nu_{\min}^{(m)}(\tau_0, \varepsilon) > 0 \qquad (\tau_0 = \varepsilon t_0 \in [0, L])$$

then the trivial solution of Eq. (3.1) cannot be stable over the finite interval $[t_0, t_0 + \Delta t]$, i.e. $\Delta t = 0$.

Theorem 3.3. If

$$\mu^{(m)}(\tau_0, \varepsilon) + \varepsilon^{m+1}\nu_{\min}^{(m)}(\tau_0, \varepsilon) \leqslant 0 \leqslant \mu^{(m)}(\tau_0, \varepsilon) + \varepsilon^{m+1}\nu_{\max}^{(m)}(\tau_0, \varepsilon)$$

$$(\tau_0 = \varepsilon t_0 \in [0, L])$$

then the trivial solution of Eq. (3.1) may not be stable on the finite interval $[t_0, t_0 + \Delta t]$.

4. Applying the results of Section 3 to Eq. (2.1), we obtain the following conditions of stability and instability of unperturbed motion:

$$\begin{array}{ll} \mu^{(m)}(t_0) + \nu_{\max}^{(m)}(t_0) < 0 & \text{is a sufficient condition for stability} \\ \mu^{(m)}(t_0) + \nu_{\min}^{(m)}(t_0) \leqslant 0 & \text{is the necessary condition for stability} \\ \mu^{(m)}(t_0) + \nu_{\min}^{(m)}(t_0) > 0 & \text{is a sufficient condition for instability} \\ \mu^{(m)}(t_0) + \nu_{\max}^{(m)}(t_0) \geqslant 0 & \text{is the necessary condition for instability} \end{array} \tag{4.1}$$

Here

$$\mu^{(m)}(t) = \mu^{(m)}(\tau, \varepsilon)|_{\varepsilon=1}, \qquad \nu_{\min,\max}^{(m)}(t) = \nu_{\min,\max}^{(m)}(\tau, \varepsilon)|_{\varepsilon=1} \tag{4.2}$$

Inequalities (4.1) constitute a complete family of necessary and sufficient conditions corresponding to the numbers $m = 0, 1, 2, \ldots, l-1$. Every m is associated with its own "strip of uncertainty",

$$-\nu_{\min}^{(m)} \geqslant \mu^{(m)} \geqslant -\nu_{\max}^{(m)} \tag{4.3}$$

For $m = 0$ the conditions (4.1) coincide exactly with simplest conditions (2.20), and strip (4.3) coincides with strip (2.21).

Taking $m = 1, 2, \ldots,$ in succession, we can expect substantial narrowing of the strip of uncertainty. Computations carried out for certain real objects showed that already for $m = 1$ the eigenvalues $\nu_{\min}^{(1)}$ and $\nu_{\max}^{(1)}$ are quite small (on the order 10^{-2}, 10^{-3}), the strip practically narrowing into a line. It is likely, therefore, that one need go no further than conditions (4.1) for $m = 1$.

We call the case where $\nu_{\min, \max}^{(m)} = 0$ and $\mu^{(m)} = 0$ "critical", since the stability problem is then unsolvable in the first approximation.

The case where $\mu^{(m)}$ satisfies (4.3) and $|\nu_{\min}^{(m)}| + |\mu_{\max}^{(m)}| \neq 0$, can be called "provisionally critical", since the possibility of solving the problem from the first approximation is not excluded.

5. Estimates of the time interval Δt during which unperturbed motion is stable, are of interest for practical purposes.

Let

$$\mu^{(m)}(t) + \nu_{\max}^{(m)}(t) < 0 \qquad (t \in [t_0, t_1] \subset [t_0, T], [t_0, L])$$
$$\mu^{(m)}(t_1) + \nu_{\max}^{(m)}(t_1) = 0 \tag{5.1}$$

The interval Δt can therefore be estimated from the inequality

$$\Delta t < t_1 - t_0 \tag{5.2}$$

Note. It is clear that unperturbed motion is uniformly stable over any finite interval $[t', t'') \subset [t_0, t_1]$, where t_1 satisfies conditions (5.1).

The interval Δt can be estimated more precisely from inequality (5.2) by using the value of t_1 given by the conditions

$$\int_{t_0}^{t} (\mu^{(m)}(t') + \nu_{\max}^{(m)}(t'))\, dt' < 0 \quad (t \in [t_0, t_1) \subset [t_0, T], \; [t_0, L])$$

$$\int_{t_0}^{t_1} (\mu^{(m)}(t') + \nu_{\max}^{(m)}(t'))\, dt' = 0$$

BIBLIOGRAPHY

1. Kamenkov, G. V., Stability of motion over a finite time interval. PMM Vol. 17, №5, 1953.
2. Lebedev, A. A., Stability of motion over a prescribed time interval. PMM Vol. 18, №2, 1954.
3. Lebedev, A. A., Using the method of "frozen coefficients" to investigate the stability of unsteady motion. Izv. VUZ, Aviatsionnaia Tekhnika №1, 1958.
4. Abgarian, K. A., Asymptotic splitting of the equations of a linear automatic control system. Dokl. Akad. Nauk SSSR Vol. 166, №2, 1966.

Translated by A.Y.